

# Generalized entropic measures of quantum correlations

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We propose a general measure of non-classical correlations for bipartite systems based on generalized entropic functions and majorization properties. Defined as the minimum information loss due to a local measurement, in the case of pure states it reduces to the generalized entanglement entropy, i.e., the generalized entropy of the reduced state. However, in the case of mixed states it can be non-zero in separable states, vanishing just for states diagonal in a general product basis, like the Quantum Discord. Simple quadratic measures of quantum correlations arise as a particular case of the present formalism. The minimum information loss due to a joint local measurement is also discussed. The evaluation of these measures in a few simple relevant cases is as well provided, together with comparison with the corresponding entanglement monotones.

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## I. INTRODUCTION

Quantum entanglement is well known to be an essential resource for performing certain quantum information processing tasks such as quantum teleportation [1, 2]. It has also been shown to be essential for achieving an exponential speed-up over classical computation in the case of pure-state based quantum computation [3]. However, in the case of mixed-state quantum computation, such as the model of Knill and Laflamme [4], such speed-up can be achieved without a substantial presence of entanglement [5]. This fact has turned the attention to other types and measures of quantum correlations, like the quantum discord (QD) [6, 7], which, while reducing to the entanglement entropy in bipartite pure states, can be non-zero in certain separable mixed states involving mixtures of non-commuting product states. It was in fact shown in [8] that the circuit of [4] does exhibit a non-negligible value of the QD between the control qubit and the remaining qubits. As a result, interest on the QD [9–13] and other alternative measures of quantum correlations for mixed states [14–17] has grown considerably.

The aim of this work is to embed measures of quantum correlations within a general formulation based on majorization concepts [2, 18, 19] and the generalized information loss induced by a measurement with unknown result. This framework is able to provide general entropic measures of quantum correlations for mixed quantum states with properties similar to those of the QD, like vanishing just for states diagonal in a standard or conditional product basis (i.e., classical or partially classical states) and reducing to the corresponding generalized entanglement entropy in the case of pure states. But as opposed to the QD and other related measures [14, 17], which are based essentially on the von Neumann entropy

$$S(\rho) = -\text{Tr } \rho \log_2 \rho, \quad (1)$$

and rely on specific associated properties, the present measures are applicable with general entropic forms satisfying minimum requirements [18, 20]. For instance, they

can be directly applied with the linear entropy

$$S_2(\rho) = 2(1 - \text{Tr } \rho^2), \quad (2)$$

which corresponds to the linear approximation  $-\ln \rho \approx 1 - \rho$  in (1) and is directly related to the purity  $\text{Tr } \rho^2$  and the pure state concurrence [21, 22], and whose evaluation in a general situation is easier than (1) as it does not require explicit knowledge of the eigenvalues of  $\rho$ . We will show, however, that the same qualitative information can nonetheless be obtained. The positivity of the QD relies on the special concavity property of the conditional von Neumann entropy [2, 6, 18], which prevents its direct extension to general entropic forms.

The concepts of generalized entropies, generalized information loss by measurement and the ensuing entropic measures of quantum correlations based on minimum information loss due to local or joint local measurements are defined and discussed in II. Their explicit evaluation in three specific examples is provided in III, where comparison with the corresponding entanglement monotones is also discussed. Conclusions are finally drawn in IV.

## II. FORMALISM

### A. Generalized entropies

Given a density operator  $\rho$  describing the state of a quantum system ( $\rho \geq 0$ ,  $\text{Tr } \rho = 1$ ), we define the generalized entropies [20]

$$S_f(\rho) = \text{Tr } f(\rho), \quad (3)$$

where  $f(p)$  is a smooth strictly concave real function defined for  $p \in [0, 1]$  satisfying  $f(0) = f(1) = 0$  ( $f$  is continuous in  $[0, 1]$  and  $f'$  strictly decreasing in  $(0, 1)$ , such that  $f(qp_i + (1-q)p_j) > qf(p_i) + (1-q)f(p_j) \forall q \in (0, 1)$  and  $p_i \neq p_j$ ). We will further assume here  $f''(p) < 0 \forall p \in (0, 1)$ , which ensures strict concavity. As in (1)–(2), we will normalize entropies such that  $S_f(\rho) = 1$  for a maximally mixed single qubit state ( $2f(1/2) = 1$ ). While our whole discussion can be directly extended to more

general concave or Schur-concave [19] functions, we will concentrate here on the simple forms (3) which already include many well known instances: The von Neumann entropy (1) corresponds to  $f(p) = -p \log_2 p$ , the linear entropy (2) to  $f(p) = 2(p - p^2)$ , and the Tsallis entropy [23]  $S_q(\rho) \propto 1 - \text{Tr } \rho^q$  to  $f(p) = (p - p^q)/(1 - 2^{1-q})$  for the present normalization, which is concave for  $q > 0$ . It reduces to the linear entropy (2) for  $q = 2$  and to the von Neumann entropy (1) for  $q \rightarrow 1$ . The Rényi entropy [18]  $S_q^R(\rho) = (\log_2 \text{Tr } \rho^q)/(1 - q)$  is just an increasing function of  $S_q(\rho)$ . The Tsallis entropy has been recently employed to derive generalized monogamy inequalities [24]. Entropies of the general form (3) were used to formulate a generalized entropic criterion for separability [25, 26], on the basis of the majorization based disorder criterion [27], extending the standard entropic criterion [28].

While additivity amongst the forms (3) holds only in the von Neumann case ( $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$ ), strict concavity and the condition  $f(0) = f(1) = 0$  ensure that all entropies (3) satisfy [20]: i)  $S_f(\rho) \geq 0$ , with  $S_f(\rho) = 0$  if and only if (iff)  $\rho$  is a pure state ( $\rho^2 = \rho$ ), ii) they are concave functions of  $\rho$  ( $S_f(\sum_i q_i \rho_i) \geq \sum_i q_i S_f(\rho_i)$  if  $q_i \geq 0$ ,  $\sum_i q_i = 1$ ) and iii) *they increase with increasing mixedness* [18]:

$$\rho' \prec \rho \Rightarrow S_f(\rho') \geq S_f(\rho), \quad (4)$$

where  $\rho' \prec \rho$  indicates that  $\rho'$  is *majorized* by  $\rho$  [18, 19]:

$$\rho' \prec \rho \Leftrightarrow \sum_{j=1}^i p'_j \leq \sum_{j=1}^i p_j, \quad i = 1, \dots, n-1. \quad (5)$$

Here  $p_i, p'_i$  denote the eigenvalues of  $\rho$  and  $\rho'$  sorted in *decreasing* order ( $p_i \geq p_{i+1}$ ,  $\sum_{i=1}^n p_i = 1$ ) and  $n = n'$  the dimension of  $\rho$  and  $\rho'$  (if different, the smaller set of eigenvalues is to be completed with zeros). Essentially  $\rho' \prec \rho$  indicates that the probabilities  $\{p'_i\}$  are more spread out than  $\{p_i\}$ . The maximally mixed state  $\rho_n = I_n/n$  satisfies  $\rho_n \prec \rho \forall \rho$  of dimension  $n$ , implying that all entropies  $S_f(\rho)$  attain their maximum at such state:  $S_f(\rho) \leq S_f(\rho_n) = n f(1/n) \forall \rho$  of rank  $r \leq n$ .

Eq. (4) follows from concavity (and the condition  $f(0) = 0$  if  $n \neq n'$ ) as for  $n = n'$ ,  $\rho' \prec \rho$  iff  $\rho'$  is a mixture unitaries of  $\rho$  [2, 18] ( $\rho' = \sum_i q_i U_i \rho U_i^\dagger$ ,  $q_i > 0$ ,  $U_i^\dagger U_i = I$ ), and  $S_f(U_i \rho U_i^\dagger) = S_f(\rho)$ . Moreover, if at least one of the inequalities in (5) is strict ( $<$ ), then  $S_f(\rho') > S_f(\rho)$ , as  $S_f(\rho) = \sum_i f(p_i)$  is a strictly decreasing function of the partial sums  $s_i = \sum_{j=1}^i p_j$  [25] ( $\partial S_f / \partial s_i = f'(p_i) - f'(p_{i+1}) < 0$  if  $p_{i+1} < p_i$ ,  $i < n$ ).

While the converse of Eq. (4) does not hold in general ( $S_f(\rho') \geq S_f(\rho) \not\Rightarrow \rho' \prec \rho$ ), it does hold if valid for *all*  $S_f$  of the present form (an example of a smooth sufficient set was provided in [26]):

$$S_f(\rho') \geq S_f(\rho) \forall S_f \Rightarrow \rho' \prec \rho. \quad (6)$$

Hence, although the rigorous concept of disorder implied by majorization ( $\rho' \prec \rho$ ) cannot be captured by any single choice of entropy, consideration of the general forms (3) warrants complete correspondence through Eq. (6).

## B. Generalized information loss by measurement

Let us now consider a general projective measurement  $M$  on the system, described by a set of orthogonal projectors  $P_k$  ( $\sum_k P_k = I_n$ ,  $P_k P_{k'} = \delta_{kk'} P_k$ ). The state of the system after this measurement, if the result is unknown, is given by [2]

$$\rho' = \sum_k P_k \rho P_k, \quad (7)$$

which is just the “diagonal” of  $\rho$  in a particular basis ( $\rho' = \sum_j |\langle j' | \rho | j' \rangle| |j'\rangle \langle j'|$ , with  $|j'\rangle$  the eigenvectors of the blocks  $P_k \rho P_k$ ). It is well known that such diagonals are always more mixed than the original  $\rho$  [18, 19], i.e.,  $\rho' \prec \rho$ , and hence, for any  $f$  of the present form,

$$S_f(\rho') \geq S_f(\rho). \quad (8)$$

Moreover,  $S_f(\rho') = S_f(\rho)$  iff  $\rho' = \rho$ , i.e., if  $\rho$  is unchanged by such measurement (if  $\rho = \sum_i p_i |i\rangle \langle i| \neq \rho'$ , strict concavity implies  $S_f(\rho') = \sum_j f(\sum_i p_i |\langle j' | i \rangle|^2) > \sum_{i,j} |\langle j' | i \rangle|^2 f(p_i) = \sum_i f(p_i)$ ). A measurement with unknown result entails then no gain and most probably a loss of information according to any  $S_f$ . The difference

$$I_f^M(\rho) = S_f(\rho') - S_f(\rho) \quad (9)$$

quantifies, according to the measure  $S_f$ , this loss of information, i.e., the information contained in the off-diagonal elements of  $\rho$  in the basis  $\{|j'\rangle\}$ . It then satisfies  $I_f^M(\rho) \geq 0$ , with  $I_f^M(\rho) = 0$  iff  $\rho' = \rho$ .

In the case of the von Neumann entropy (1), Eq. (9) reduces to the *relative* entropy [2, 18, 29] between  $\rho$  and  $\rho'$ , since their diagonal elements in the basis  $\{|j'\rangle\}$  coincide:

$$I^M(\rho) = S(\rho') - S(\rho) \quad (10a)$$

$$= \text{Tr } \rho (\log_2 \rho - \log_2 \rho') = S(\rho || \rho'). \quad (10b)$$

The relative entropy  $S(\rho || \rho')$  is well known to be non-negative  $\forall \rho, \rho'$ , vanishing just if  $\rho = \rho'$  [2, 18]. In the case of the linear entropy (2), Eq. (9) becomes instead

$$I_2^M(\rho) = 2\text{Tr}(\rho^2 - \rho'^2) \quad (11a)$$

$$= 2\text{Tr} \rho(\rho - \rho') = 2\|\rho - \rho'\|^2, \quad (11b)$$

where  $\|A\| = \sqrt{\text{Tr} A^\dagger A}$  is the Hilbert-Schmidt or Frobenius norm. Hence,  $I_2^M(\rho)$  is just the square of the norm of the off-diagonal elements in the measured basis, being again verified that  $I_2^M(\rho) = 0$  only if  $\rho' = \rho$ .

Let us remark, however, that the general positivity of (9) arises just from the majorization  $\rho' \prec \rho$  and the strict concavity of  $S_f$ , the specific properties of the measures (10b)–(11b) being not invoked. In fact, if the off-diagonal elements of  $\rho$  in the measured basis are sufficiently small, a standard perturbative expansion of (9) shows that

$$I_f^M(\rho) \approx \sum_{j < k} \frac{f'(p'_k) - f'(p'_j)}{p'_j - p'_k} |\langle j' | \rho | k' \rangle|^2, \quad (12)$$

where  $p'_j = \langle j' | \rho | j' \rangle$ . The fraction in (12) is positive  $\forall p'_j \neq p'_k$  due to the concavity of  $f$  (if  $p'_j = p'_k$ , it should be replaced by  $-f''(p'_j) < 0$ ). Eq. (12) is just the square of a *weighted* quadratic norm of the off-diagonal elements. In the case (2), Eq. (12) reduces of course to Eq. (11b).

For generalized measurements [2] leading to

$$\rho' = \sum_k M_k \rho M_k^\dagger, \quad (13)$$

Eq. (8) and the positivity of (9) remain valid  $\forall S_f$  if *both* conditions i)  $\sum_k M_k^\dagger M_k = I$  and ii)  $\sum_k M_k M_k^\dagger = I$  are fulfilled: if  $|j'\rangle$  and  $|i\rangle$  denote the eigenvectors of  $\rho'$  and  $\rho$ , we then have  $\sum_{j,k} |\langle j' | M_k | i \rangle|^2 = \sum_{i,k} |\langle j' | M_k | i \rangle|^2 = 1$  and hence  $S_f(\rho') = \sum_j f(\sum_{k,i} |\langle j' | M_k | i \rangle|^2 p_i) \geq \sum_{j,k,i} |\langle j' | M_k | i \rangle|^2 f(p_i) = \sum_i f(p_i)$ , i.e.,  $\rho' \prec \rho$ . While i) ensures trace conservation, ii) warrants that the eigenvalues of  $\rho'$  are convex combinations of those of  $\rho$ . If not valid, Eq. (8) no longer holds in general, as already seen in trivial single qubit examples ( $M_0 = |0\rangle\langle 0|$ ,  $M_1 = |0\rangle\langle 1|$  will change any state  $\rho$  into the pure state  $|0\rangle\langle 0|$ , yet fulfilling i)). For projective measurements,  $M_k = P_k$ .

### C. Minimum information loss by a local measurement

Let us now consider a bipartite system  $A + B$  whose state is specified by a density matrix  $\rho_{AB}$ . Suppose that a complete *local* measurement  $M_B$  in system  $B$  is performed, defined by one dimensional local projectors  $P_j^B = |j_B\rangle\langle j_B|$ . The state after this measurement (Eq. (7) with  $P_k \rightarrow I_A \otimes P_j^B$ ) becomes

$$\rho'_{AB} = \sum_j q_j \rho_{A/j} \otimes P_j^B, \quad (14)$$

where  $q_j = \text{Tr}[\rho_{AB} I_A \otimes P_j^B]$  is the probability of outcome  $j$  and  $\rho_{A/j} = \text{Tr}_B[\rho_{AB} I_A \otimes P_j^B] / q_j$  the reduced state of  $A$  after such outcome. The quantity

$$I_f^{M_B}(\rho_{AB}) = S_f(\rho'_{AB}) - S_f(\rho_{AB}) \quad (15)$$

will quantify the ensuing loss of information.

We can now define the minimum of Eq. (15) amongst all such measurements, which will depend just on  $\rho_{AB}$ :

$$I_f^B(\rho_{AB}) = \min_{M_B} I_f^{M_B}(\rho_{AB}). \quad (16)$$

Eq. (8) implies  $I_f^B(\rho_{AB}) \geq 0$ , with  $I_f^B(\rho_{AB}) = 0$  iff there is a complete local measurement in  $B$  which leaves  $\rho_{AB}$  unchanged, i.e., if  $\rho_{AB}$  is already of the form (14). These states are in general diagonal in a *conditional* product basis  $\{|i_j j\rangle \equiv |i_j^A\rangle \otimes |j_B\rangle\}$ , where  $\{|i_j^A\rangle\}$  is the set of eigenvectors of  $\rho_{A/j}$ , and can be considered as *partially* classical, as there is a local measurement in  $B$  (but not necessarily in  $A$ ) which leaves them unchanged. They are the same states for which the QD vanishes [6, 7]. Eq.

(16) can then be considered a measure of the deviation of  $\rho$  from such states, i.e., of quantum correlations. One may similarly define  $I_f^A(\rho_{AB})$  as the minimum information loss due to a local measurement in system  $A$ , which may differ from  $I_f^B(\rho_{AB})$ .

The states (14) are *separable* [30], i.e., convex superpositions of product states ( $\rho_{AB}^s = \sum_\alpha q_\alpha \rho_A^\alpha \otimes \rho_B^\alpha$ ,  $q_\alpha > 0$ ). Nonetheless, for a general  $\rho_{AB}^s$  the different terms  $\rho_A^\alpha \otimes \rho_B^\alpha$  may not commute, in contrast with (14). Hence, Eq. (16) will be positive not only in entangled (i.e., unseparable) states, but also in all separable states not of the form (14), detecting those quantum correlations emerging from the mixture of non-commuting product states.

Eq. (14) and concavity imply the basic bound  $S_f(\rho'_{AB}) \geq \sum_j q_j S_f(\rho_{A/j})$ . In addition, we also have the less trivial lower bounds

$$I_f^B(\rho_{AB}) \geq S_f(\rho_A) - S_f(\rho_{AB}), \quad (17a)$$

$$I_f^B(\rho_{AB}) \geq S_f(\rho_B) - S_f(\rho_{AB}), \quad (17b)$$

where  $\rho_{A,B} = \text{Tr}_{B,A} \rho_{AB}$  are the local reduced states. The r.h.s. in (17) is negative or zero in any separable state [25, 27], but can be positive in an entangled state. Proof: *Any* separable state is more disordered globally than locally [27], as in a classical system [18]:  $\rho_{AB}^s \prec \rho_A^s$ ,  $\rho_{AB}^s \prec \rho_B^s$  [27], or equivalently [25],  $S_f(\rho_{AB}^s) \geq S_f(\rho_A^s)$ ,  $S_f(\rho_{AB}^s) \geq S_f(\rho_B^s) \forall f$ . For the state (14) this implies

$$S_f(\rho'_{AB}) \geq S_f(\rho'_A) = S_f(\rho_A), \quad (18a)$$

$$S_f(\rho'_{AB}) \geq S_f(\rho'_B) \geq S_f(\rho_B), \quad (18b)$$

since  $\rho'_A = \text{Tr}_B \rho'_{AB} = \sum_j q_j \rho_{A/j} = \rho_A$ , while  $\rho'_B = \text{Tr}_A \rho'_{AB} = \sum_j q_j P_j^B$  is just the diagonal of the actual  $\rho_B$  in the basis determined by the local projectors  $P_j^B$  and hence  $\rho'_B \prec \rho_B$ . Eqs. (18) lead then to Eqs. (17). The same inequalities (17) hold of course for  $I_f^A(\rho_{AB})$ .

One may be tempted to choose as the optimal local measurement which minimizes Eq. (15) that based on the eigenvectors of the reduced state  $\rho_B$ , in which case it will remain unchanged after measurement ( $\rho'_B = \rho_B$ ). Although this choice is optimal in the case of pure states (see IID) and other relevant situations (see III), it may not be so for a general  $\rho_{AB}$ . For instance, even if local states are maximally mixed, the optimal local measurement may not be arbitrary (see example 3 in III). In such a case a minor perturbation can orientate the local eigenstates along any preferred direction, different from that where the lost information is minimum.

### D. Pure states and generalized entanglement entropy

If  $\rho_{AB}$  is pure ( $\rho_{AB}^2 = \rho_{AB}$ ), then

$$I_f^B(\rho_{AB}) = I_f^A(\rho_{AB}) = S_f(\rho_A) = S_f(\rho_B), \quad (19)$$

i.e., Eq. (16) reduces to the generalized entropy of the subsystem (*generalized entanglement entropy*), quantifying the entanglement between  $A$  and  $B$  according to the

measure  $S_f$ . In the von Neumann case (1), Eq. (19) becomes the standard entanglement entropy [31]  $E_{AB} = S(\rho_A) = S(\rho_B)$ , whereas in the case of the linear entropy (2), Eq. (19) becomes the square of the *pure state concurrence* (i.e., the tangle) [22],  $C_{AB}^2 = S_2(\rho_A) = S_2(\rho_B)$ . Proof: For a pure state  $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ ,  $S_f(\rho_{AB}) = 0$  and both  $\rho_A, \rho_B$  have the same non-zero eigenvalues. Eqs. (17) then imply  $I_f^B(\rho_{AB}) \geq S_f(\rho_A) = S_f(\rho_B)$ . There is also a local measurement which saturates Eqs. (17): It is that determined by the Schmidt decomposition

$$|\Psi_{AB}\rangle = \sum_{k=1}^{n_s} \sqrt{p_k} |k_s^A\rangle \otimes |k_s^B\rangle, \quad (20)$$

where  $n_s$  is the Schmidt number and  $p_k$  the non-zero eigenvalues of  $\rho_A$  or  $\rho_B$  [2]. Choosing the local projectors in (14) as  $P_k^B = |k_s^B\rangle\langle k_s^B|$ , we then obtain

$$\rho'_{AB} = \sum_k p_k P_k^A \otimes P_k^B, \quad (21)$$

which leads to local states  $\rho'_A = \rho_A = \sum_k p_k P_k^A$ ,  $\rho'_B = \rho_B = \sum_k p_k P_k^B$  and hence to

$$S_f(\rho'_{AB}) = S_f(\rho_A) = S_f(\rho_B) = \sum_k f(p_k), \quad (22)$$

implying Eq. (19). For pure states, entanglement can then be considered as *the minimum information loss due to a local measurement*, according to *any*  $S_f$ .

Just to verify Eq. (19), we note that for an arbitrary local measurement defined by projectors  $P_j^B = |j_B\rangle\langle j_B|$ , we may rewrite Eq. (20) as

$$|\Psi_{AB}\rangle = \sum_j \sqrt{q_j} |\Psi_{A/j}\rangle \otimes |j_B\rangle, \quad (23)$$

where  $|\Psi_{A/j}\rangle = \sum_k \sqrt{p_k/q_j} |j_B|k_s^B\rangle |k_s^A\rangle$  and  $q_j = \sum_k p_k |\langle j_B|k_s^B\rangle|^2$ , such that  $\rho_{A/j} = |\Psi_{A/j}\rangle\langle\Psi_{A/j}|$  in (14). Hence, by concavity  $S_f(\rho'_{AB}) = \sum_j f(q_j) \geq \sum_k f(p_k) \forall S_f$ , i.e.,  $\{q_j\} \prec \{p_k\}$ . Thus, for pure states, a local measurement in the basis where  $\rho_B$  is diagonal (local Schmidt basis) provides the minimum of Eq. (15)  $\forall S_f$ . For a maximally entangled state leading to a maximally mixed  $\rho_B$  ( $p_k = 1/n_B \forall k$ ) Eq. (15) becomes obviously independent of the choice of local basis (any choice in  $B$  leads to a corresponding basis in  $A$ , leaving (20) unchanged).

A pure state  $|\Psi_{AB}^I\rangle$  can be said to be *absolutely* more entangled than another pure state  $|\Psi_{AB}^{II}\rangle$  if  $S_f(\rho_A^I) \geq S_f(\rho_A^{II}) \forall S_f$ , i.e., if  $\rho_A^I \prec \rho_A^{II}$  ( $\{p_k^I\} \prec \{p_k^{II}\}$ ). This concept has a clear deep implication: According to the theorem of Nielsen [32], a pure state  $|\Psi_{AB}^I\rangle$  can be obtained from  $|\Psi_{AB}^{II}\rangle$  by local operations and classical communication (LOCC) only if  $\rho_A^I \prec \rho_A^{II}$ , i.e., iff  $|\Psi_{AB}^I\rangle$  is absolutely more entangled than  $|\Psi_{AB}^{II}\rangle$ . This condition cannot be ensured by a single choice of entropy, requiring the present general measures for an entropic formulation (the exception being two-qubit or  $2 \times d$  systems, where

any  $S_f(\rho_A)$  is a decreasing function of the largest eigenvalue  $p_1$  of  $\rho_A$  and hence  $S_f(\rho_A^I) \geq S_f(\rho_A^{II})$  iff  $\rho_A^I \prec \rho_A^{II}$ ).

The *convex roof extension* [22, 33] of the generalized entanglement entropy (19) of pure states will lead to an entanglement measure for mixed states,

$$E_f(\rho_{AB}) = \min_{\sum_{\alpha} q_{\alpha} \rho_{AB}^{\alpha} = \rho_{AB}} \sum_{\alpha} q_{\alpha} E_f(\rho_{AB}^{\alpha}), \quad (24)$$

where  $q_{\alpha} > 0$ ,  $\rho_{AB}^{\alpha} = |\Psi_{AB}^{\alpha}\rangle\langle\Psi_{AB}^{\alpha}|$  are pure states and  $E_f(\rho_{AB}^{\alpha}) = S_f(\rho_A^{\alpha})$  is the generalized entanglement entropy of  $|\Psi_{AB}^{\alpha}\rangle$ . Minimization is over all representations of  $\rho_{AB}$  as convex combinations of pure states. Eq. (24) is a non-negative quantity which clearly vanishes iff  $\rho_{AB}$  is separable. It is also an *entanglement monotone* [33] (i.e., it cannot increase by LOCC) since  $E_f(\rho_{AB}^{\alpha})$  is a concave function of  $\rho_A^{\alpha}$  invariant under local unitaries, satisfying then the conditions of ref. [33]. In the case of the von Neumann entropy, Eq. (24) becomes the entanglement of formation (EOF)  $E(\rho_{AB})$  [34], while in the case of the linear entropy, it leads to the mixed state tangle  $\tau(\rho_{AB})$  [22, 35]. The general mixed state concurrence  $C(\rho_{AB})$  [22] (denoted there as  $I$ -concurrence) is recovered for  $E_f(\rho_{AB}^{\alpha}) = \sqrt{S_2(\rho_A^{\alpha})}$  ( $\tau = C^2$  in two qubit systems [35], but not necessarily in general).

While  $I_f^B(\rho_{AB}) = 0$  implies  $E_f(\rho_{AB}) = 0$  (as (14) is separable) the converse is not true since  $I_f^B(\rho_{AB})$  can be non-zero in separable states. Nonetheless, and despite coinciding for pure states, there is no general order relation between these two quantities for a general  $\rho_{AB}$ .

### E. Minimum information loss by a joint local measurement

We now consider the information loss  $I_f^{M_{AB}}(\rho_{AB})$  due to a measurement  $M_{AB}$  based on *products*  $P_i^A \otimes P_j^B$  of one dimensional local projectors, such that  $\rho'_{AB}$  is the diagonal of  $\rho_{AB}$  in a *standard* product basis  $\{|ij\rangle = |i_A\rangle \otimes |j_B\rangle\}$ :

$$\rho'_{AB} = \sum_{i,j} p_{ij} P_i^A \otimes P_j^B, \quad (25)$$

where  $p_{ij} = \langle ij|\rho_{AB}|ij\rangle$ . Such measurement can be considered as a subsequent local measurement in  $A$  after a measurement in  $B$  (if the results are of course unknown), implying  $I_f^{M_{AB}}(\rho_{AB}) \geq I_f^{M_B}(\rho_{AB})$ , where  $M_B = \{P_j^B\}$  is the measurement in  $B$ . The ensuing minimum

$$I_f^{AB}(\rho_{AB}) = \min_{M_{AB}} I_f^{M_{AB}}(\rho_{AB}), \quad (26)$$

will then satisfy in general

$$I_f^{AB}(\rho_{AB}) \geq I_f^B(\rho_{AB}), \quad (27)$$

with  $I_f^{AB}(\rho_{AB}) = 0$  if and only if  $\rho_{AB}$  is of the form (25). The state (25) represents a *classically correlated state* [14, 36]. For such states there is a local measurement

in  $A$  as well as in  $B$  which leaves the state unchanged, being equivalent in this product basis to a classical system described by a joint probability distribution  $p_{ij}$ . Eq. (26) is then a measure of all quantum-like correlations. The states (25) are of course a particular case of (14), i.e., that where all  $\rho_{A/j}$  are mutually commuting. Product states  $\rho_A \otimes \rho_B$  are in turn a particular case of (25) ( $p_{ij} = p_i^A p_j^B \forall i, j$ ) and correspond to  $\rho_{A/j}$  independent of  $j$  in (14).

In the case of pure states we obtain, however,

$$I_f^{AB}(\rho_{AB}) = I_f^B(\rho_{AB}) = S_f(\rho_A) = S_f(\rho_B), \quad (28)$$

since the state (21) is already of the form (25), being left unchanged by a measurement based on the Schmidt basis projectors  $P_{k'}^A \otimes P_k^B$ . Pure state entanglement can then be also seen as *the minimum information loss due to a joint local measurement*.

For an arbitrary product measurement on a pure state, the expansion

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij} |i_A\rangle \otimes |j_B\rangle, \quad (29)$$

with  $c_{ij} = \sum_k \sqrt{p_k} \langle i_A | k_s^A \rangle \langle j_B | k_s^B \rangle$ , leads to  $p_{ij} = |c_{ij}^2|$  in (25). Eqs. (25)–(28) then imply  $S_f(\rho'_{AB}) = \sum_{i,j} f(|c_{ij}^2|) \geq \sum_k f(p_k) \forall S_f$ . Since  $I_f^{M_{AB}}(\rho_{AB}) \geq I_f^{M_B}(\rho_{AB}) \geq I_f^B(\rho_{AB})$ , Eqs. (23), (28) and (29) lead to

$$\{|c_{ij}^2|\} \prec \{q_j\} \prec \{p_k\}. \quad (30)$$

The first relation is apparent as  $q_j = \sum_i |c_{ij}^2|$  is just the marginal of the joint distribution  $|c_{ij}^2|$ . The state (21) can then be rigorously regarded as *the closest classical state* to the pure state  $\rho_{AB}$ , since it provides the lowest information loss among *all* local or joint local measurements for *any*  $S_f$ . Pure states have therefore an associated *least mixed classical state*, such that the state obtained after any local measurement is always majorized by it.

Let us finally mention that it is also feasible to consider more general product measurements  $M_{A/B}$  based on conditional product projectors  $P_{i_j}^A \otimes P_j^B$ , leading to a  $\rho'_{AB}$  diagonal in a conditional product basis,

$$\rho'_{AB} = \sum_{i,j} p_{ij} P_{i_j}^A \otimes P_j^B \quad (31)$$

where  $p_{ij} = \langle i_j j | \rho_{AB} | i_j j \rangle$ . The ensuing information loss will satisfy again  $I_f^{M_{A/B}}(\rho_{AB}) \geq I_f^{M_B}(\rho_{AB})$ , as (31) can still be considered as the diagonal of (14) in a conditional product basis  $\{|i_j j\rangle\}$ , where the  $\{|i_j^A\rangle\}$  are not necessarily the eigenvectors of  $\rho_{A/j}$ . However, if chosen as the latter, we have  $I_f^{M_{A/B}}(\rho_{AB}) = I_f^{M_B}(\rho_{AB})$  and hence,

$$I_f^{A/B}(\rho_{AB}) = \text{Min}_{M_{A/B}} I_f^{M_{A/B}}(\rho_{AB}) = I_f^B(\rho_{AB}), \quad (32)$$

as (14) remains unchanged under a measurement in the optimum conditional product basis formed by the eigenvectors of the  $\rho_{A/j}$  times the states  $|j_B\rangle$ .

## F. Von Neumann based measures

If  $S_f(\rho)$  is chosen as the von Neumann entropy (1), Eq. (15) becomes (see Eq. (10b))

$$I^{M_B}(\rho_{AB}) = S(\rho'_{AB}) - S(\rho_{AB}) = S(\rho_{AB} || \rho'_{AB}). \quad (33)$$

The ensuing minimum  $I^B(\rho_{AB})$  is also the *minimum* relative entropy between  $\rho_{AB}$  and any state  $\rho'_{AB}$  diagonal in a standard or conditional product basis:

$$I^B(\rho_{AB}) = \text{Min}_{M_B} I^{M_B}(\rho_{AB}) = \text{Min}_{\rho'_{AB}} S(\rho_{AB} || \rho'_{AB}), \quad (34)$$

where  $\rho'_{AB}$  denotes a state of the general form (14) with *both* the local projectors  $P_j^B = |j_B\rangle\langle j_B|$  as well as the probabilities  $q_j$  and states  $\rho_{A/j}$  being arbitrary.

Proof: For a given choice of conditional product basis, the minimum relative entropy is obtained when  $\rho'_{AB}$  has the same diagonal elements as  $\rho_{AB}$  in that basis (as  $-\sum_i p_i \log_2 q_i$  is minimized for  $q_i = p_i$ ). Hence,  $S(\rho_{AB} || \rho'_{AB}) \geq S(\rho_{AB} || \rho'_{AB}) = I^{M_{A/B}}(\rho_{AB}) \geq I^B(\rho_{AB})$ , where  $\rho'_{AB}$  denotes here the post-measurement state (31) in that basis.

The same property holds for  $I^{AB}(\rho_{AB})$  if  $\rho'_{AB}$  is restricted to states diagonal in a standard product basis:

$$I^{AB}(\rho_{AB}) = \text{Min}_{M_{AB}} I^{M_{AB}}(\rho_{AB}) = \text{Min}_{\rho'_{AB}} S(\rho_{AB} || \rho'_{AB}), \quad (35)$$

where  $\rho'_{AB}$  is here of the form (25) with  $p_{ij}$  arbitrary. Eq. (35) is precisely the bipartite version of the quantity  $D$  introduced in [17] as a measure of quantum correlations for composite systems.

The quantity (34) is also closely related to the *quantum discord* [6–8], which can be written in the present notation as  $D^B(\rho_{AB}) = \text{Min}_{M_B} D^{M_B}(\rho_{AB})$ , with

$$\begin{aligned} D^{M_B}(\rho_{AB}) &= S(\rho'_{AB}) - S(\rho'_B) - [S(\rho_{AB}) - S(\rho_B)] \\ &= I^{M_B}(\rho_{AB}) - I^{M_B}(\rho_B), \end{aligned} \quad (36)$$

where  $\rho'_{AB}$  is the measured state (14) and  $\rho'_B$ ,  $\rho_B$  the reduced states after and before the measurement. Thus,  $D^B(\rho_{AB}) \leq I^B(\rho_{AB})$ . They will coincide when the optimal local measurement is the same for both (15) and (36) and corresponds to the basis where  $\rho_B$  is diagonal, such that  $\rho'_B = \rho_B$  ( $I^{M_B}(\rho_B) = 0$ ). This coincidence takes place, for instance, whenever  $\rho_B$  is maximally mixed (as in this case  $\rho'_B = \rho_B$  for any choice of local basis). Both  $D^B(\rho_{AB})$  and  $I^B(\rho_{AB})$  also vanish for the same type of states (i.e., those of the form (14)) and both reduce to the standard entanglement entropy  $E_{AB} = S(\rho_A)$  for pure states (although Eq. (16) requires a measurement in the local Schmidt basis whereas (36) becomes independent of the choice of local basis, as  $\rho_{A/j}$  is pure and hence  $S(\rho'_{AB}) = S(\rho'_B)$  for any local measurement). A direct generalization of (36) to a general entropy  $S_f(\rho)$  is no longer positive for a general concave  $f$ , since the positivity of (36) relies on the concavity of the *conditional* von Neumann entropy  $S(A|B) = S(\rho_{AB}) - S(\rho_B)$  [18], which does not hold for a general  $S_f$ .

Minimum distances between  $\rho_{AB}$  and classical states of the form (25) were also considered in [14], where the attention was focused on the decrease  $Q$  of the mutual information  $S(\rho_A) + S(\rho_B) - S(\rho_{AB})$  after a measurement  $M_{AB}$  in the product basis formed by the eigenstates of  $\rho_A$  and  $\rho_B$ . Such quantity coincides with present  $I^{M_{AB}}(\rho_{AB})$  for this choice of basis as  $\rho_A$  and  $\rho_B$  remain unchanged. Nonetheless, for a general  $\rho_{AB}$  the minimum (35) may be attained at a different basis.

### G. Quadratic measure

If  $S_f(\rho)$  is chosen as the linear entropy (2), Eq. (15) becomes (see Eq. (11b))

$$I_2^{M_B}(\rho_{AB}) = 2\text{Tr}(\rho_{AB}^2 - \rho_{AB}'^2) = 2\|\rho_{AB} - \rho_{AB}'\|^2, \quad (38)$$

where  $\|\rho_{AB} - \rho_{AB}'\|^2 = \sum_{j \neq j', i, k} |\langle ij | \rho_{AB} | kj' \rangle|^2$  is just the squared norm of the off-diagonal elements lost after the local measurement. It therefore provides the simplest measure of the information loss. Its minimum is the *minimum* squared Hilbert-Schmidt distance between  $\rho_{AB}$  and any state  $\rho_{AB}^d$  diagonal in a general product basis:

$$I_2^B(\rho_{AB}) = \min_{M_B} I_2^{M_B}(\rho_{AB}) = \min_{\rho_{AB}^d} \|\rho_{AB} - \rho_{AB}^d\|^2, \quad (39)$$

where the last minimization is again over all states of the form (14), with  $P_j^B$ ,  $q_j$  and  $\rho_{A/j}$  arbitrary.

Proof: For a general product basis,  $\|\rho_{AB} - \rho_{AB}^d\|^2 = \|\rho_{AB} - \rho_{AB}'\|^2 + \|\rho_{AB}' - \rho_{AB}^d\|^2$ , where  $\rho_{AB}'$  is again the diagonal of  $\rho_{AB}$  in this basis. Hence, the optimum choice in this basis is  $\rho_{AB}^d = \rho_{AB}'$ , whence  $\|\rho_{AB} - \rho_{AB}^d\|^2 \geq \|\rho_{AB} - \rho_{AB}'\|^2 = I_2^{M_{A/B}}(\rho_{AB}) \geq I_2^B(\rho_{AB})$ . Actually, we could also extend the last minimization in (39) to all operators  $O_{AB}^d$  diagonal in a general product basis.

The same property holds for  $I_2^{AB}(\rho_{AB})$  if  $\rho_{AB}^d$  is restricted to states diagonal in a standard product basis:

$$I_2^{AB}(\rho_{AB}) = \min_{M_{AB}} I_2^{M_{AB}}(\rho_{AB}) = \min_{\rho_{AB}^d} \|\rho_{AB} - \rho_{AB}^d\|^2, \quad (40)$$

where  $\rho_{AB}^d$  is here of the general form (25). Note that  $I_2^{M_{AB}}(\rho_{AB}) = I_2^{M_B}(\rho_{AB}) + \sum_{j, i \neq k} |\langle ij | \rho_{AB} | kj \rangle|^2$  is just the squared norm of all off-diagonal elements.

In the case of pure states, Eqs. (39) and (40) reduce to the pure state concurrence [22]  $C_{AB}^2 = S_2(\rho_A)$ .

### III. EXAMPLES

We will now evaluate the general measures (16) and (26) for any  $S_f$  in a few simple relevant examples.

#### A. Mixture of a general pure state with the maximally mixed state

For a convex mixture of  $|\Psi_{AB}\rangle = \sum_{k=1}^{n_s} \sqrt{p_k} |k_s^A k_s^B\rangle$  (Eq. (20)) with the maximally mixed state, i.e.,

$$\rho_{AB}(x) = x|\Psi_{AB}\rangle\langle\Psi_{AB}| + \frac{1-x}{n} I_A \otimes I_B, \quad (41)$$

where  $x \in [0, 1]$  and  $n = n_A n_B$ , the minimum  $I_f^B(x) \equiv I_f^B[\rho_{AB}(x)]$  corresponds again to a measurement in the local Schmidt basis for  $|\Psi_{AB}\rangle$  and is given by

$$I_f^B(x) = \sum_{k=1}^{n_s} [f(xp_k + \frac{1-x}{n}) - f(\delta_{k1}x + \frac{1-x}{n})], \quad (42)$$

with  $I_f^A(x) = I_f^{AB}(x) = I_f^B(x)$ . Eq. (42) is a *strictly increasing function* of  $x \forall S_f$  if  $n_s \geq 2$  (i.e., if  $|\Psi_{AB}\rangle$  is entangled), implying  $I_f^B(x) > 0 \forall x \in (0, 1]$ .

Proof: After a local measurement in the basis  $\{|k_s^B\rangle\}$ , the joint state becomes

$$\rho_{AB}'(x) = x \sum_{k=1}^{n_s} p_k P_k^A \otimes P_k^B + \frac{1-x}{n} I_A \otimes I_B, \quad (43)$$

which is diagonal in the Schmidt basis  $\{|k_s^A\rangle \otimes |k_s^B\rangle\}$  with diagonal elements  $p_{k'k} = \delta_{k'k}xp_k + \frac{1-x}{n}$ . For any other complete local measurement,  $\rho_{AB}'$  will be diagonal in a basis  $\{|i_j^A\rangle \otimes |j_B\rangle\}$ , where we set  $|j_B^A\rangle = |\Psi_{A/j}\rangle$  (Eq. (23)), with diagonal elements  $p_{ij}^A = \delta_{ij}xq_j + \frac{1-x}{n}$ . The latter are always majorized by  $p_{kk'}$  ( $\{p_{ij}^A\} \prec \{p_{kk'}\}$ ) since  $\{q_j\} \prec \{p_k\}$  (Eq. (30)) and  $x \geq 0$ . Hence,  $S_f(\rho_{AB}')$  is minimum for a measurement in the basis  $\{|k_s^B\rangle\}$ , which leads to Eq. (42). Moreover,  $I_f^{AB}(x) = I_f^A(x) = I_f^B(x)$  since (43) is diagonal in a standard product basis.

Eq. (43) is again the *closest* classical state to (41), majorizing *any* other state obtained after a local or product measurement.

To verify the monotonicity, we note that

$$\begin{aligned} \frac{dI_f^B}{dx} &= \sum_{k=1}^{n_s} [(p_k - \frac{1}{n})f'(p_k^x) - (\delta_{k1} - \frac{1}{n})f'(\lambda_k^x)] \\ &\geq (\frac{n'_s - 1}{n} + \sum_{p_k < 1/n} p_k)[f'(\lambda_2^x) - f'(\lambda_1^x)] \geq 0, \end{aligned} \quad (44)$$

since  $\lambda_2^x = \frac{1-x}{n} \leq p_k^x \leq \lambda_1^x = x + \frac{1-x}{n}$  and hence  $f'(\lambda_2^x) \geq f'(p_k^x) \geq f'(\lambda_1^x)$ , where  $p_k^x = xp_k + \frac{1-x}{n}$  and  $n'_s \geq 1$  is the number of Schmidt probabilities  $p_k$  not less than  $1/n$ . Eq. (42) is then strictly increasing if  $f$  is strictly concave and  $n_s \geq 2$ , implying  $I_f^B(x) = 0$  only if  $x = 0$  or  $n_s = 1$ .

A series expansion of (42) around  $x = 0$  shows that

$$I_f^B(x) = -\frac{1}{2}x^2 f''(\frac{1}{n})(1 - \sum_k p_k^2) + O(x^3), \quad (45)$$

in agreement with Eq. (12), indicating a *universal quadratic increase* of  $I_f^B(x)$  for small  $x$  ( $f''(1/n) < 0$ ).

For the quadratic measure (38) we obtain in fact a simple quadratic dependence  $\forall x \in [0, 1]$ :

$$I_2^B(x) = x^2 I_2^B(1) = 2x^2(1 - \sum_k p_k^2). \quad (46)$$

Hence, for  $|\Psi_{AB}\rangle$  entangled,  $I_f^B(x) > 0$  as soon as the mixture (41) departs from the maximally mixed state. In contrast, any entanglement measure, like the monotones (24) or the negativity [37], requires a *finite* threshold value  $x_c > 0$ , since Eq. (41) is *separable* for small  $x$ : Any bipartite state  $\rho$  is separable if  $\text{Tr}(\rho - I_n/n)^2 \leq \frac{1}{n(n-1)}$  [38], which ensures here separability for  $x \leq \frac{1}{n-1} \leq x_c$  ( $n \geq 4$ ). In the maximally entangled case  $p_k = 1/d$ , with  $n_A = n_B = d$ , (41) is in fact separable iff  $x \leq 1/(d+1)$  [22, 39]. In general, the negativity will be positive for  $x > x_c = \frac{1}{1+n\sqrt{p_1 p_2}}$ , sorting the  $p_k$  in decreasing order.

Let us finally notice that given two pure states  $|\Psi_{AB}^I\rangle$  and  $|\Psi_{AB}^{II}\rangle$ , the ensuing mixtures (41) will satisfy, at fixed  $x \in (0, 1]$ ,  $I_f^{BI}(x) \geq I_f^{BII}(x) \forall S_f$  iff  $|\Psi_{AB}^I\rangle$  is *absolutely* more entangled than  $|\Psi_{AB}^{II}\rangle$  ( $\{p_k^I\} \prec \{p_k^{II}\}$ ). This is apparent as  $S_f(\rho_{AB}^I(x)) = S_f(\rho_{AB}^{II}(x))$  whereas  $\rho_{AB}^I(x) \prec \rho_{AB}^{II}(x)$  iff  $\{p_k^I\} \prec \{p_k^{II}\}$  (Eq. (43)), in which case  $S_f(\rho_{AB}^I(x)) \geq S_f(\rho_{AB}^{II}(x))$ .

## B. Two-qubit case

Let us now explicitly consider the mixture (41) in the two-qubit case, where  $|\Psi_{AB}\rangle$  can be always written as

$$|\Psi_{AB}\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle, \quad (47)$$

with  $|ij\rangle \equiv |i_s^A\rangle \otimes |j_s^B\rangle$  and  $p \in [0, 1]$ . For a local spin measurement along an axis forming an angle  $\theta$  with the  $z$  axis, it is easy to show that the information loss is

$$I_f^{\theta B}(x) = \sum_{\nu=\pm} [f(\frac{1+x(1+2\nu \cos \theta(2p-1))}{4}) - f(\frac{1+2\nu x}{4})]. \quad (48)$$

It is verified that for  $p \neq 1/2$ ,  $I_f^{\theta B}(x)$  is minimum for  $\theta = 0$ , i.e., for a measurement in the local Schmidt basis for  $|\Psi_{AB}\rangle$  (as  $\rho'(\theta) \prec \rho'(0)$ ), while for  $p = 1/2$  (Bell state)  $I_f^{\theta B}(x)$  is  $\theta$ -independent, as the local Schmidt basis becomes arbitrary. The minimum becomes then

$$I_f^B(x) = f(\frac{1+x(4p-1)}{4}) + f(\frac{1+x(3-4p)}{4}) - f(\frac{1+3x}{4}) - f(\frac{1-x}{4}) \quad (49)$$

(Eq. (42)), being a strictly increasing function of  $x$  if  $f$  is strictly concave and  $p \in (0, 1)$  (if  $p = 0$  or  $1$ ,  $|\Psi_{AB}\rangle$  is separable and  $I_f^B(x) = 0 \forall x$ ). It is also a decreasing function of  $p$  for  $p \in [\frac{1}{2}, 1]$  at fixed  $x$ .

In particular, for  $S_f(\rho) = S_2(\rho)$ , Eq. (49) becomes

$$I_2^B(x) = 4x^2 p(1-p). \quad (50)$$

We may compare (50) with the corresponding entanglement monotone (24) (the tangle), which coincides here

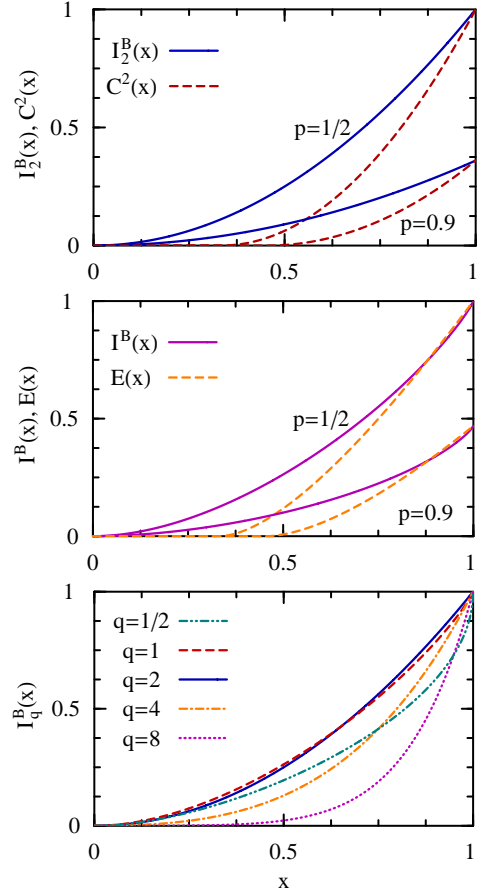


FIG. 1. Measures of quantum correlations and entanglement for the mixture (41) of the pure state (47) with the maximally mixed state, for  $p = 1/2$  and  $p = 0.9$ . Top: The quadratic measure  $I_2^B$  (Eqs. (39), (50)) and the squared concurrence  $C^2$  (Eq. (51)), satisfying  $I_2^B(x) > C^2(x)$  for  $x \in (0, 1)$ . Center: The von Neumann based measure  $I^B$  (Eqs. (34), (49) for  $f(p) = -p \log_2 p$ ) and the entanglement of formation  $E$ , again coincident for  $x = 1$  but exhibiting no fixed order relation for  $x \in (0, 1)$ . Bottom: Behavior of  $I_q^B$  (Eq. (49) for  $f(p) = \frac{p-p^q}{1-2^{1-q}}$ ) for different  $q$  and  $p = 1/2$ .  $I_1^B = I^B$  is the von Neumann measure while  $I_2^B$  the quadratic measure.

with the squared concurrence [21, 22]  $C^2(x)$  of  $\rho_{AB}(x)$ . For a general two-qubit mixed state the concurrence can be calculated as [21]  $C = \text{Max}[2\lambda_M - \text{Tr}R, 0]$ , where  $\lambda_M$  is the largest eigenvalue of  $R = \sqrt{\rho_{AB}^{1/2} \tilde{\rho}_{AB} \rho_{AB}^{1/2}}$ , with  $\tilde{\rho}_{AB} = \sigma_y^A \otimes \sigma_y^B \rho_{AB}^* \sigma_y^A \otimes \sigma_y^B$ . This leads here to

$$C(x) = \text{Max}[2x\sqrt{p(1-p)} - \frac{1-x}{2}, 0], \quad (51)$$

which vanishes for  $x \leq x_c = \frac{1}{1+4\sqrt{p(1-p)}}$ . It is then verified that for the present mixture,

$$I_2^B(x) \geq C^2(x),$$

$\forall p, x$ , with  $I_2^B(x) = C^2(x)$  just for  $x = 0$  or  $x = 1$  if  $p \in (0, 1)$ , as seen in the top panel of Fig. 1 (such inequality does not hold for any two-qubit mixed state).

In contrast, the von Neumann based measure  $I^B(x)$  (Eq. (34)) is not an upper bound to the EOF  $E(x)$  of  $\rho_{AB}(x)$ , as seen in the central panel, even though they both coincide for  $x = 1 \forall p$ . For any two qubit state,  $E$  can be evaluated in terms of the concurrence  $C$  as [21]

$$E = \sum_{\nu=\pm} f\left(\frac{1+\nu\sqrt{1-C^2}}{2}\right), \quad (52)$$

for  $f(p) = -p \log_2 p$ , which is just the relation between  $S_f(\rho)$  and  $S_2(\rho) = C^2$  for a single qubit state  $\rho$ . Hence, for  $x$  close to 1,  $E(x) - I^B(x) \approx -\frac{1-x}{4} \log_2(1-x) > 0$ , as  $E(x)$  decreases linearly whereas  $I^B(x)$  decreases logarithmically. Notice that  $I^B(x)$  coincides with the QD  $\forall p, x$ , as Eq. (36) is also minimized by a measurement along the  $z$  axis ( $\theta = 0$ ), in which case  $\rho'_B = \rho_B$ .

The bottom panel depicts the behavior of Eq. (49) for the Tsallis case  $f(p) = f_q(p) \equiv \frac{p-p^q}{1-2^{1-q}}$ . As  $q$  increases above 2,  $I_q^B(x)$  becomes less sensitive to weak quantum correlations (as  $f_q''(1/n)$  in (45) becomes small), resembling the behavior of the entanglement measures.

One may here ask if it is also possible to employ Eq. (52) with a general  $f$  for evaluating the corresponding generalized EOF (24). According to the arguments of [21] and [24], this is feasible provided Eq. (52), which is a strictly increasing function of  $C \forall$  concave  $f$ , is also *convex*. In the Tsallis case  $f(p) = f_q(p)$ , this allows the applicability of (52) for  $\frac{5-\sqrt{13}}{2} < q < \frac{5+\sqrt{13}}{2}$  (as obtained from the condition  $E''(C) \geq 0 \forall C \in [0, 1]$ ), i.e.,  $0.7 \lesssim q \lesssim 4.3$ , in agreement with the numerical results of [24]. Denoting the ensuing quantity as  $E_q(x)$ , we then obtain, for the present normalization,

$$E_2 = E_3 = C^2, \quad (53)$$

as for any single qubit state  $\rho$ ,  $S_2(\rho) = S_3(\rho) = 4 \det(\rho)$ .

The inequality  $I_q^B(x) \geq E_q(x) \forall x \in [0, 1]$  will then hold in a certain *finite* interval around  $q = 2$ , namely  $1.27 \lesssim q \lesssim 3.5$  for  $p = 1/2$  and  $1.3 \lesssim q \lesssim 4.3$  for  $p = 0.9$ . These boundaries are actually determined by the slope condition  $I_q^{B'}(1) < E_q'(1)$ . For instance, for  $p = 1/2$  and a general entropic  $f$  such that (52) is convex, we have

$$I_f^B(x) \approx 1 - \frac{1}{4}[f'(0) + 2f'(\frac{1}{2}) - 3f'(1)](1-x), \quad (54)$$

$$E_f(x) \approx 1 + \frac{3}{4}f''(\frac{1}{2})(1-x), \quad (55)$$

for  $x \rightarrow 1$ , such that  $I_f^B(x) > E_f(x)$  in this limit iff  $f'(0) + 2f'(1/2) - 3f'(1) < -3f''(1/2)$ . This leaves out the von Neumann entropy ( $f'(0) \rightarrow \infty$ ) as well as all  $q < 1$  in the Tsallis case, leading in the latter to the previous interval  $1.27 \lesssim q \lesssim 3.5$ .

### C. Decoherence of a Bell state

Let us now consider the state

$$\begin{aligned} \rho_{AB}(z) &= \frac{1}{2}[|00\rangle\langle 00| + |11\rangle\langle 11| + z(|00\rangle\langle 11| + |11\rangle\langle 00|)] \\ &= \frac{1+z}{2}|\Psi_+\rangle\langle \Psi_+| + \frac{1-z}{2}|\Psi_-\rangle\langle \Psi_-|, \end{aligned} \quad (56)$$

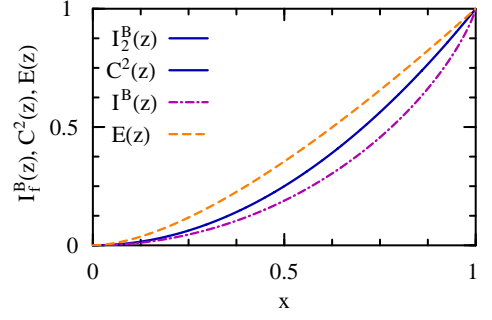


FIG. 2. Same details as fig. 1 for the state (56). Here  $I_2^B(z) = C^2(z)$  whereas  $E(z) \geq I^B(z)$  for  $z \in (0, 1)$ .

where  $|z| \leq 1$  and  $|\Psi_{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ . It corresponds to the partial decoherence of  $|\Psi_{\pm}\rangle$  and can be also seen as a mixture of these two Bell states. Even though the reduced states  $\rho_A$  and  $\rho_B$  are maximally mixed  $\forall z$ , a local spin measurement along an axis forming an angle  $\theta$  with the  $z$  axis leads to a post-measurement state  $\rho'(\theta)$  with two-fold degenerate eigenvalues  $\frac{1 \pm \sqrt{1 - \sin^2 \theta (1 - z^2)}}{4}$  and hence, to a  $\theta$ -dependent information loss

$$I_f^{\theta_B}(z) = \sum_{\nu=\pm} [2f\left(\frac{1+\nu\sqrt{1-\sin^2 \theta (1-z^2)}}{4}\right) - f\left(\frac{1+\nu z}{2}\right)]. \quad (57)$$

Its minimum for any  $z \in (-1, 1)$  and concave  $f$  corresponds again to  $\theta = 0$ , as  $\rho'(\theta) \prec \rho'(0) = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \forall \theta$ . We then obtain, setting again  $2f(1/2) = 1$ ,

$$I_f^B(z) = 1 - f\left(\frac{1+z}{2}\right) - f\left(\frac{1-z}{2}\right), \quad (58)$$

with  $I_f^{AB}(z) = I_f^A(z) = I_f^B(z)$  as  $\rho(0)$  is diagonal in a standard product basis. Hence,  $I_f^B(z) > 0$  if  $z \neq 0$ , with  $I_f^B(z) = -\frac{1}{4}f''(\frac{1}{2})z^2 + O(z^3)$  for  $z \rightarrow 0$ . Moreover,  $I_f^B(z)$  is an increasing ( $I_f^{B'}(z) > 0$ ) convex ( $I_f^{B''}(z) > 0$ ) function of  $z \forall S_f$ .

In the case of the linear entropy, Eq. (58) becomes

$$I_2^B(z) = z^2 = C^2(z), \quad (59)$$

where  $C(z) = |z|$  is the concurrence of (56). Thus, here  $I_2^B(z)$  and  $E_2(z)$  coincide exactly  $\forall z \in [0, 1]$ . In contrast, the von Neumann measure  $I^B(z)$  is *smaller* than the EOF  $E(z) = \sum_{\nu=\pm} f\left(\frac{1+\nu\sqrt{1-z^2}}{2}\right)$  ( $f(p) = -p \log_2 p$ )  $\forall z \in (0, 1)$  (Fig. 2). For small  $z$  we have in particular  $E(z) \approx -\frac{1}{2}z^2 \log_2 z^2 > I^B(z) \approx \frac{1}{2}z^2 / \ln 2$ . Again,  $I^B(z)$  coincides here with the QD as  $\rho_B$  is maximally mixed.

Let us finally remark that Eqs. (53) and (58) also imply

$$I_3^B(z) = z^2 = E_3(z).$$

It can then be seen that for  $2 < q < 3$ ,  $I_q^B(z) > E_q(z) \forall z \in (0, 1)$  (although the difference is small) whereas for  $q < 2$  or  $q > 3$  (within the limits allowed by the validity of (52))  $I_q^B(z) < E_q(z) \forall z \in (0, 1)$ . These intervals can be corroborated from the expansions for  $z \rightarrow 0$  and  $z \rightarrow 1$ ,

$$\begin{aligned} I_f^B(z) - E_f(z) &= \frac{1}{4}[-f''(\frac{1}{2}) - f'(0) + f'(1)]z^2 + O(z^3), \\ &= \frac{1}{4}[-f''(\frac{1}{2}) - f'(0) + f'(1)](1-z) + O(1-z)^2 \end{aligned}$$



which imply  $I_f^B(z) > E_f(z)$  in these limits iff  $f'(0) - f'(1) < -f''(1/2)$ , leading to  $2 < q < 3$  in Tsallis case.

#### IV. CONCLUSION

We have constructed a general entropic measure of quantum correlations  $I_f^B(\rho_{AB})$ , which represents the minimum loss of information, according to the entropy  $S_f$ , due to a local projective measurement. Its basic properties are similar to those of the quantum discord, vanishing for the same partially classical states (14) and coinciding with the corresponding generalized entanglement entropy in the case of pure states. Its positivity relies, however, entirely on the majorization relations fulfilled by the post-measurement state, being hence applicable with general entropic forms based on arbitrary concave functions. In particular, for the linear entropy it leads to a quadratic measure  $I_2^B(\rho_{AB})$  which is particularly simple to evaluate and can be directly interpreted as minimum squared distance, yet providing the same qualitative information as other measures. The minimum loss of information due to a joint local measurement  $I_f^{AB}(\rho_{AB})$ , has also been discussed, and shown to coincide with  $I_f^B(\rho_{AB})$

in some important situations, vanishing just for the classically correlated states (25).

While there is no general order relation between these quantities and the associated entanglement monotones (24), the use of generalized entropies allows at least to find such a relation in some particular cases: The quadratic measure  $I_2^B(\rho_{AB})$  provides for instance an upper bound to the squared concurrence of the two-qubit states (41)–(47) (unlike the von Neumann based measures) and coincides with it in the mixture (56). Moreover, generalized entropies such as  $S_q(\rho)$  allow to find in these previous cases an interval of  $q$  values where an order relationship holds, which requires a delicate balance between the derivatives of  $f$  at different points.

Let us finally mention that some general concepts emerge naturally from the present formalism, like that of absolutely more entangled and in particular that of the *least mixed* classically correlated state that can be associated with certain states, such as pure states or the mixtures (41) or (56). This state majorizes any other state obtained after a local measurement, thus minimizing the entropy increase (15) or (26) for *any* choice of entropy  $S_f$ . It allows for an unambiguous identification of the least perturbing local measurement.

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